Class 10/11/2016

Note Title 9/29/2016

Cauchy integral formula

Theorem:

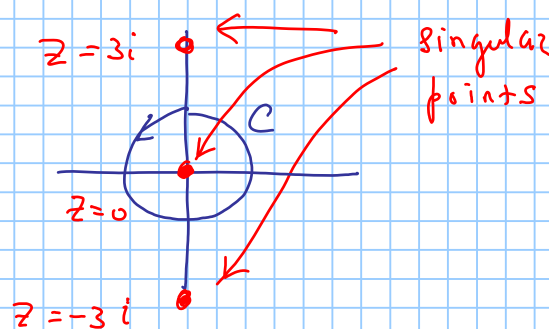
Let f be analytic everywhere inside and on a simple closed contour C taked with counterclockwise orientation.

If z_0 is interiour point of C then:

$$\int_C \frac{f(z)}{(z-z_0)}\,dz = 2\pi i\, f(z_0)$$

Example:

$$\int_{|z|=1} \frac{\cos z}{z(z^2+9)}\,dz$$



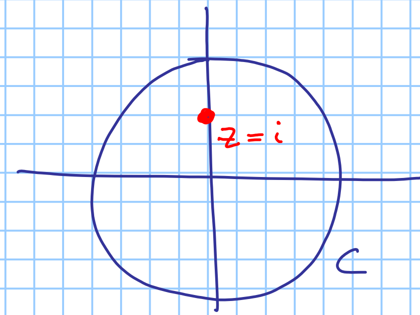
$$= \int_{|z|=0} \frac{\cos(z)/(z^2+9)}{z}\,dz$$

$\cos(z)/(z^2+9)$ – analytic inside C $\Rightarrow$

$$= 2\pi i \quad \cos(0)/(0+9) = \boxed{\frac{2\pi i}{9}}$$

Example 2

$$\frac{1}{2\pi i} \int_{|z|=2} \frac{z e^z}{(z-i)}\, dz$$



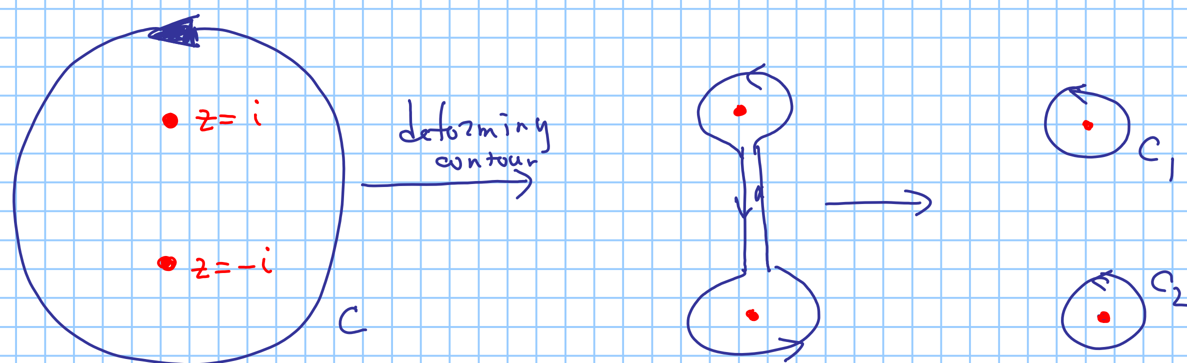
$z e^z$ – entire function

$\Rightarrow$ analytic inside C $\Longrightarrow$

$$\frac{1}{2\pi i}\left(i e^{i} \cdot 2\pi i \right) = \boxed{i e^{i}}$$

Example 3:

$$\int_{|z|=2} \frac{z e^z}{1+z^2}\, dz$$



$$\Rightarrow \int_{|z|=2} \frac{z e^z}{1+z^2}\, dz = \int_{C_1} \frac{z e^z}{(1+z^2)}\, dz + \int_{C_2} \frac{z e^z}{(1+z^2)}\, dz$$

1 $$\int_{C_1} \frac{z e^z}{(1+z^2)}\, dz = \int_{C_1} \frac{z e^z/(z+i)}{(z-i)}\, dz$$

$\frac{z e^z}{(z+i)}$ – analytic inside C_1 $\Longrightarrow$

$$= \frac{i e^{i}}{(i+i)} \cdot 2\pi i = \pi i e^{i}$$

Similarly

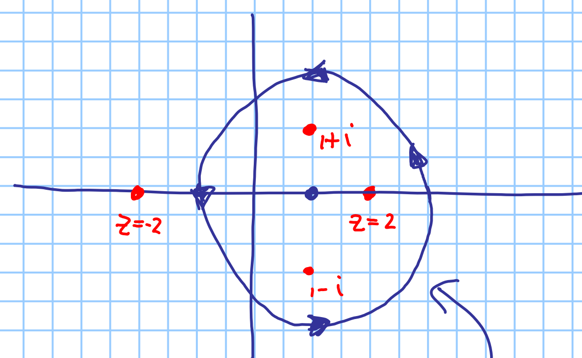
$$\int_{C_2} \frac{z e^z}{(1+z^2)} dz = \int_{C_2} \frac{z e^z/(z-i)}{(z+i)} dz = 2\pi i \frac{(-i)e^{-i}}{-2i}$$

$$= \pi i e^{-i}$$

$$\Rightarrow \int \frac{z e^z}{1+z^2} dz = \pi i e^{i} + \pi i e^{-i} = 2\pi i \frac{e^{i}+e^{-i}}{2} = \boxed{2\pi i \cos(1)}$$

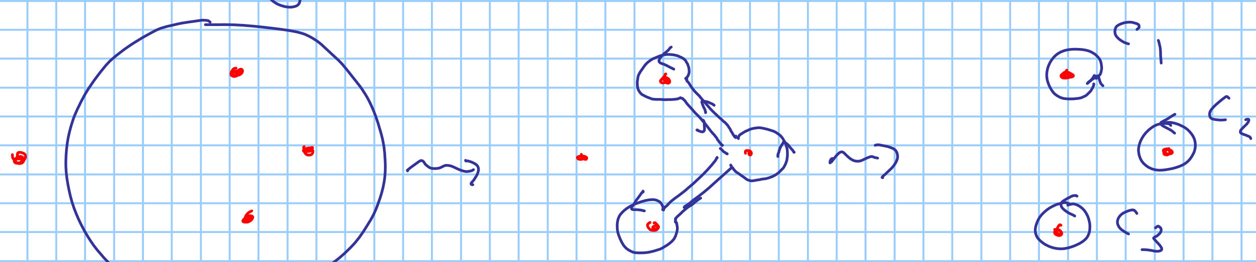
Example 4:

$$\int_{|z-1|=2} \frac{dz}{(z^2-4)(z^2-2z+2)}$$



$$\frac{1}{(z^2-4)(z^2-2z+2)} = \frac{1}{(z-2)(z+2)(z-(1+i))(z-(1-i))}$$

Deforming contour:



$\Rightarrow$

(1) $$\int_{C_1} \frac{dz}{(z-2)(z+2)(z-(1+i))(z-(1-i))} = \frac{2\pi i}{(i-1)(i+3)(2i)}$$

$$= \frac{\pi}{-1-3-i+3i} = \frac{\pi}{2i-4} = \frac{\pi(-4-2i)}{20} =$$

$$= \boxed{-\frac{\pi}{5} - \frac{\pi i}{10}}$$

(2) $$\int_{C_2} \frac{dz}{(z-2)(z+2)(z-(i+1))(z-(1-i))} = \frac{2\pi i}{4(1-i)(1+i)} =$$

$$= \frac{\pi i}{2(2)} = \boxed{\frac{\pi i}{4}}$$

(3) $$\int_{C_3} \frac{dz}{(z-2)(z+2)(z-(1+i))(z-(1-i))} = \frac{2\pi i}{(-1-i)(3-i)(-2i)}$$

$$= \frac{\pi}{5} - \frac{\pi i}{10}$$

$\Rightarrow$

$$(1) + (2) + (3) = \cancel{-\frac{\pi}{5}} - \frac{\pi i}{10} + \frac{\pi i}{4} + \cancel{\frac{\pi}{5}} - \frac{\pi i}{10} =$$

$$= \frac{5\pi i - 4\pi i}{20} = \boxed{\frac{\pi i}{20}}$$

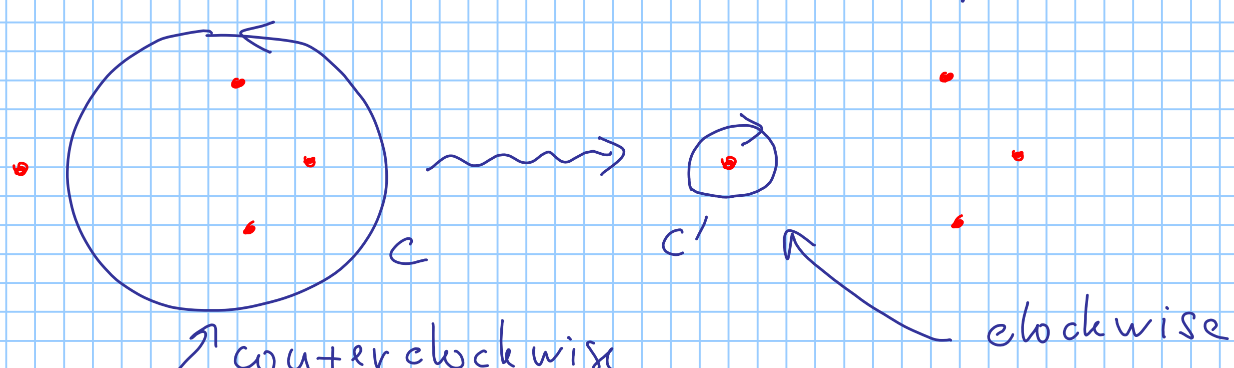
__Example:__ The same integral deforming contour to the point $z=-2$

$$\frac{1}{(z^2-4)(z^2-2z+2)} \xrightarrow{z=\frac{1}{w}} \frac{w^4}{(1-4w^2)(2w^2-2w+1)}$$

This function is analytic at $w=0$

$\Rightarrow$ original function analityc at $z=\infty$

$\Rightarrow$ We can deform contour trough ∞-point



$$\Rightarrow \int_{C'} \frac{dz}{(z-2)(z+2)(z-(1+i))(z-(1-i))} =$$

$$= \frac{2\pi i}{(-2-2)(-3-i)(-3+i)} = \frac{2\pi i}{(-4)(9+1)}$$

$$= -\frac{\pi i}{20}$$

Finally, contour has opposite orientation

$\Rightarrow$ multiply by (-1) to obtain:

$$\boxed{\frac{\pi i}{20}} \longleftarrow$$

Same answer as in previous example.